§5. $\Lambda\alpha$ LS potential

5.1. *Quark-model RGM kernel*

In order to generate the $\Lambda\alpha$ LS potential, we use $\boldsymbol{P}=0$ Wigner transform of the ΛN quark-model exchange kernel just as a local ΛN LS potential:

$$v_{\Lambda N}^{LS}=\sum_{\mathcal{T}}\Big\{X_{\mathcal{T}}^{LS}G_{W\mathcal{T}}^{LS}(\boldsymbol{R},0)(\boldsymbol{L}\cdot\boldsymbol{S})+X_{\mathcal{T}}^{LS^{(-)}}G_{W\mathcal{T}}^{LS}(\boldsymbol{R},0)(\boldsymbol{L}\cdot\boldsymbol{S}^{(-)}) \\ +X_{\mathcal{T}}^{LS^{(-)}\sigma}G_{W\mathcal{T}}^{LS}(\boldsymbol{R},0)(\boldsymbol{L}\cdot\boldsymbol{S}^{(-)})P_\sigma\Big\}\quad , \tag{5·1}$$

where

$$X_{\mathcal{T}}^{\Omega}=\left(X_{\mathcal{T}}^{\Omega}\right)^{ud}+\left(X_{\mathcal{T}}^{\Omega}\right)^{s}\mathcal{P}' \tag{5·2}$$

is the ΛN-ΛN component of the spin-flavor factors for the three different LS types, $\Omega=LS$, $LS^{(-)}$ and $LS^{(-)}\sigma$. Here we assume the flavor exchange operator $\mathcal{P}'$ operates on the flavor wave function of the ket side. We move $\mathcal{P}'$ to the right most place and replace it with $\mathcal{P}'\to -P_\sigma P_r$. If we take the matrix element of Eq. (5·1) with respect to the spin-flavor functions of the $\Lambda\alpha$ system, the nucleon spin operator part disappears due to the spin saturated property of the α cluster and we obtain the spin-flavor part as

$$\boldsymbol{L}\cdot 4\left[\left(X_{\mathcal{T}}^{LS}\right)^{ud}+\left(X_{\mathcal{T}}^{LS^{(-)}}\right)^{ud}\right]\boldsymbol{S}_\Lambda-\boldsymbol{L}P_r\cdot 4\left[\left(X_{\mathcal{T}}^{LS}\right)^{s}+\left(X_{\mathcal{T}}^{LS^{(-)}\sigma}\right)^{s}\right]\boldsymbol{S}_\Lambda\quad , \tag{5·3}$$

where $\boldsymbol{S}_\Lambda$ is the spin operator of the Λ particle. We therefore only need to calculate the spatial part of $G_{W\mathcal{T}}^{LS}(\boldsymbol{R},0)\boldsymbol{L}$ and $G_{W\mathcal{T}}^{LS}(\boldsymbol{R},0)\boldsymbol{L}P_r$. For this calculation, we can also use the formula Eq. (2.33) or (by changing $\boldsymbol{P}\to\boldsymbol{P}+3\boldsymbol{q}/4$)

$$V(\boldsymbol{q}_f,\boldsymbol{q}_i)=e^{-\frac{3}{8\nu}\frac{\boldsymbol{k}^2}{4}}\left(\frac{2}{3\pi\nu}\right)^{\frac{3}{2}}\int d\boldsymbol{P}e^{-\frac{2}{3\nu}\boldsymbol{P}^2}u\left(\boldsymbol{k},\frac{\zeta+4}{4(\zeta+1)}\boldsymbol{q}-\frac{\zeta}{\zeta+1}\boldsymbol{P};\boldsymbol{P}+\frac{3}{4}\boldsymbol{q}\right). \tag{5·4}$$

For a simple Gaussian function, the momentum representation for $u=e^{-\kappa r^2}\boldsymbol{L}$ and $u=-e^{-\kappa r^2}\boldsymbol{L}P_r$ in Eq. (2.35) are calculated as

$$u(\boldsymbol{k}',\boldsymbol{q}';\boldsymbol{P})=\left(\frac{\pi}{\kappa}\right)^{\frac{3}{2}}e^{-\frac{1}{4\kappa}\boldsymbol{k}'^2}i\frac{1}{2\kappa}\left[\boldsymbol{q}'\times\boldsymbol{k}'\right]\qquad\text{for}\qquad u=e^{-\kappa r^2}\boldsymbol{L}\quad , \\ u(\boldsymbol{k}',\boldsymbol{q}';\boldsymbol{P})=\left(\frac{\pi}{\kappa}\right)^{\frac{3}{2}}e^{-\frac{1}{\kappa}\left(\boldsymbol{q}'+\frac{1}{2}\frac{\zeta-1}{\zeta+1}\boldsymbol{P}\right)^2}i\frac{1}{2\kappa}\left[\left(\boldsymbol{q}'+\frac{1}{2}\frac{\zeta-1}{\zeta+1}\boldsymbol{P}\right)\times\left(\boldsymbol{k}'-\frac{\zeta-1}{\zeta+1}\boldsymbol{P}\right)\right] \\ \text{for}\qquad u=-e^{-\kappa r^2}\boldsymbol{L}P_r\quad . \tag{5·5}$$

From these we can easily obtain the following results for $V_d^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i)i\boldsymbol{n}$ and $V_e^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i)i\boldsymbol{n}$ with $\boldsymbol{n}=[\boldsymbol{q}\times\boldsymbol{k}]$, which is the LS version of Eq. (2.39) :

$$V_d^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i)=\left(\frac{\pi}{\kappa}\right)^{\frac{3}{2}}\exp\left\{-\frac{1}{4}\left(\frac{3}{8\nu}+\frac{1}{\kappa}\right)\boldsymbol{k}^2\right\}\frac{1}{2\kappa}\frac{\zeta+4}{4(\zeta+1)}\qquad\text{for}\qquad u(r)=e^{-\kappa r^2}\boldsymbol{L}$$

Table I. The LS spin-flavor factors in the ΛN-ΛN channel. This can be read from our first LS paper, Phys. Lett. B318 (1993) 587 - 591.

$\mathcal{T}$	$\left(X_{\mathcal{T}}^{LS}\right)^{ud}$	$\left(X_{\mathcal{T}}^{LS}\right)^{s}$	$\left(X_{\mathcal{T}}^{LS^{(-)}}\right)^{ud}$	$\left(X_{\mathcal{T}}^{LS^{(-)}\sigma}\right)^{s}$
D_-	$\frac{1}{9}$	$\frac{1}{18}\left(1+\frac{4}{\lambda}+\frac{1}{\lambda^2}\right)$	$-\frac{1}{9}$	$-\frac{1}{18}\left(1-\frac{1}{\lambda^2}\right)$
D_+	$-\frac{1}{108}\left(1+\frac{8}{\lambda}+\frac{3}{\lambda^2}\right)$	0	$\frac{1}{108}\left(1-\frac{4}{\lambda}-\frac{3}{\lambda^2}\right)$	0
S, S'	$\frac{1}{216}\left(1-\frac{4}{\lambda}+\frac{3}{\lambda^2}\right)$	$\frac{1}{36}\left(1+\frac{2}{\lambda}-\frac{1}{\lambda^2}\right)$	$-\frac{1}{216}\left(1+\frac{8}{\lambda}-\frac{3}{\lambda^2}\right)$	$-\frac{1}{36}\left(1-\frac{1}{\lambda}\right)^2$

$$V_e^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i)=\left(\frac{8\pi}{3}\frac{1}{\nu+\frac{8}{3}\kappa}\right)^{\frac{3}{2}}\exp\left\{-\frac{3}{8\nu}\frac{\boldsymbol{k}^2}{4}-\frac{25}{24}\frac{1}{\nu+\frac{8}{3}\kappa}\boldsymbol{q}^2\right\}\frac{5}{6}\frac{1}{\nu+\frac{8}{3}\kappa}$$
$$\text{for}\qquad u(r)=e^{-\kappa r^2}\boldsymbol{L}P_r \tag{5·6}$$

After all, the $\Lambda\alpha$ LS potential for the Gaussian LS force with $u(r)=e^{-\kappa r^2}$ is given by

$$V_{\Lambda\alpha}^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i)=\left[X_d\ V_d^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i)+X_e\ V_e^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i)\right]i\boldsymbol{n}\cdot\boldsymbol{S}_\Lambda \tag{5·7}$$

with

$$X_d=4\left[\left(X_{\mathcal{T}}^{LS}\right)^{ud}+\left(X_{\mathcal{T}}^{LS^{(-)}}\right)^{ud}\right]\quad,\qquad X_e=4\left[\left(X_{\mathcal{T}}^{LS}\right)^{s}+\left(X_{\mathcal{T}}^{LS^{(-)}\sigma}\right)^{s}\right]\quad. \tag{5·8}$$

We calculate these spatial integrals for each of the interaction types, $\mathcal{T}=D_-$, D_+ and S (or S'). From our previous paper [Phys. Lett. B318 (1993) 587 - 591], we find

$$u(r)=-\sqrt{\frac{2}{\pi}}\alpha_S x^3 m_{ud}c^2\left(\frac{3}{2}\right)^3\exp\left\{-\frac{3}{4}\left(\frac{r}{b}\right)^2\right\}$$
$$\times\begin{Bmatrix}\frac{1}{\sqrt{6}}\\ \frac{3}{\sqrt{2}}h_1\left(\frac{9}{4}\left(\frac{r}{b}\right)^2\right)\\ \frac{3}{4}h_1\left(\frac{9}{32}\left(\frac{r}{b}\right)^2\right)\end{Bmatrix}\quad\text{for}\quad\begin{Bmatrix}D_-\\ D_+\\ S\quad\text{or}\quad S'\end{Bmatrix}\quad, \tag{5·9}$$

where

$$h_1(a)=3\int_0^1 e^{-at^2}t^2dt\quad. \tag{5·10}$$

(We have changed the old notation $h_1(x^2)$ to $h_1(x)$.) The spin-flavor factor can be read from the same paper as in Table I. Thus Eq. (5·8) gives the result in Table II. Note that the most important knock-on term of the D_- type gives a zero contribution because of the cancellation of the LS and $LS^{(-)}$ factors in the up-down sector. If the quark mass ratio, $\lambda=m_s/m_{ud}$, goes to the infinity, all of these spin-flavor factors vanish, which is a well known property of the spin-flavor SU_6 wave function of the Λ particle. (Only the strange quark of Λ contributes to the spin-related quantities like the magnetic moment,

Table II. The spin-flavor factors of the $\Lambda\alpha$ potential for the quark-model iteration.

$\mathcal{T}$	X_d	X_e
D_-	0	$\frac{4}{9\lambda}\left(2+\frac{1}{\lambda}\right)$
D_+	$-\frac{2}{9\lambda}\left(2+\frac{1}{\lambda}\right)$	0
S, S'	$-\frac{1}{9\lambda}\left(2-\frac{1}{\lambda}\right)$	$\frac{2}{9\lambda}\left(2-\frac{1}{\lambda}\right)$

since the up-down di-quarks are coupled in the spin-isospin zero.) The spatial integrals for Eq. (5·9) are calculated from the formula in Eq. (5·6). After some calculations, we obtain

$$
\begin{aligned}
V_d^{D_+}(\boldsymbol{q}_f,\boldsymbol{q}_i) &= -2\pi\left(\frac{3}{4}\right)^{\frac{3}{2}}\alpha_S x^3 m_{ud}c^2b^5\frac{\zeta+4}{4(\zeta+1)}\exp\left\{-\frac{1}{4}\left(\frac{3}{8\nu}+\frac{b^2}{3}\right)\boldsymbol{k}^2\right\}\widetilde{h}_1\left(\frac{1}{2}b|\boldsymbol{k}|\right) \quad , \\
V_d^{S}(\boldsymbol{q}_f,\boldsymbol{q}_i) &= -2\pi\left(\frac{12}{11}\right)^{\frac{3}{2}}\alpha_S x^3 m_{ud}c^2b^5\frac{\zeta+4}{4(\zeta+1)}\exp\left\{-\left(\frac{1}{4}\frac{3}{8\nu}+\frac{8}{33}b^2\right)\boldsymbol{k}^2\right\} \\
&\quad\times\widetilde{h}_1\left(\frac{1}{\sqrt{11}}b|\boldsymbol{k}|\right) \quad , \\
V_e^{D_-}(\boldsymbol{q}_f,\boldsymbol{q}_i) &= -\frac{5\pi}{4}\alpha_S x^3 m_{ud}c^2b^5\left(\frac{1}{1+\frac{1}{2}b^2\nu}\right)^{\frac{5}{2}}\exp\left\{-\frac{3}{32\nu}\boldsymbol{k}^2-\frac{25}{48}\frac{(b\boldsymbol{q})^2}{1+\frac{1}{2}b^2\nu}\right\} \quad , \\
V_e^{S}(\boldsymbol{q}_f,\boldsymbol{q}_i) &= -\frac{5\pi}{4}\alpha_S x^3 m_{ud}c^2b^5\left(\frac{12}{11+4b^2\nu}\right)^{\frac{3}{2}}\frac{1}{1+\frac{1}{2}b^2\nu}\exp\left\{-\frac{3}{32\nu}\boldsymbol{k}^2-\frac{25}{6}\frac{(b\boldsymbol{q})^2}{11+4b^2\nu}\right\} \\
&\quad\times\widetilde{h}_1\left(\frac{1}{\sqrt{\left(1+\frac{1}{2}b^2\nu\right)\left(11+4b^2\nu\right)}}\frac{5}{4}b|\boldsymbol{q}|\right) \quad , \qquad (5\cdot11)
\end{aligned}
$$

where

$$
\widetilde{h}_1(x) = 3e^{-x^2}\int_0^1 e^{x^2t^2}t^2dt \quad . \qquad (5\cdot12)
$$

Combining Eq. (5·11) and Table II, we finally obtain

$$
V_{\Lambda\alpha}^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i) = F(q_f,q_i,\theta)\; i\boldsymbol{n}\cdot\boldsymbol{S}_\Lambda \qquad (5\cdot13)
$$

with

$$
\begin{aligned}
&F(q_f,q_i,\theta) = 2\pi\alpha_S x^3 m_{ud}c^2b^5\exp\left\{-\frac{3}{32\nu}\boldsymbol{k}^2\right\} \\
&\times\left\{\frac{\zeta+4}{4(\zeta+1)}\left[\frac{2}{9\lambda}\left(2+\frac{1}{\lambda}\right)\left(\frac{3}{4}\right)^{\frac{3}{2}}\exp\left\{-\frac{1}{12}(b\boldsymbol{k})^2\right\}\widetilde{h}_1\left(\frac{1}{2}b|\boldsymbol{k}|\right)\right.\right. \\
&\left.\left.+\frac{2}{9\lambda}\left(2-\frac{1}{\lambda}\right)\left(\frac{12}{11}\right)^{\frac{3}{2}}\exp\left\{-\frac{8}{33}(b\boldsymbol{k})^2\right\}\widetilde{h}_1\left(\frac{1}{\sqrt{11}}b|\boldsymbol{k}|\right)\right]\right.
\end{aligned}
$$

$$-\frac{5}{8}\frac{1}{1+\frac{1}{2}b^2\nu}\left[\frac{4}{9\lambda}\left(2+\frac{1}{\lambda}\right)\left(\frac{1}{1+\frac{1}{2}b^2\nu}\right)^{\frac{3}{2}}\exp\left\{-\frac{25}{48}\frac{(b\boldsymbol{q})^2}{1+\frac{1}{2}b^2\nu}\right\}\right.$$
$$+\frac{4}{9\lambda}\left(2-\frac{1}{\lambda}\right)\left(\frac{12}{11+4b^2\nu}\right)^{\frac{3}{2}}\exp\left\{-\frac{25}{6}\frac{(b\boldsymbol{q})^2}{11+4b^2\nu}\right\}$$
$$\left.\times\widetilde{h}_1\left(\frac{5}{4}\frac{b|\boldsymbol{q}|}{\sqrt{\left(1+\frac{1}{2}b^2\nu\right)(11+4b^2\nu)}}\right)\right]\Bigg\} \quad . \tag{5·14}$$

Here, $\boldsymbol{k}=\boldsymbol{q}_f-\boldsymbol{q}_i$ and $\boldsymbol{q}=(\boldsymbol{q}_f+\boldsymbol{q}_i)/2$. The partial wave components defined through

$$V^{LS}_{\Lambda\alpha}(\boldsymbol{q}_f,\boldsymbol{q}_i)=4\pi\sum_{JM\ell}V^{LS}_{\ell J}(q_f,q_i)\mathcal{Y}_{(\ell\frac{1}{2})JM}(\widehat{\boldsymbol{q}}_f,\mathrm{spin})\mathcal{Y}^*_{(\ell\frac{1}{2})JM}(\widehat{\boldsymbol{q}}_i,\mathrm{spin}) \tag{5·15}$$

are calculated from the formula in Appendix C of our LS RGM paper [Prog. Theor. Phys. 103 (2000) 755 - 794]:

$$V^{LS}_{\ell J}(q_f,q_i)=\frac{1}{2}\int_{-1}^{1}F(q_f,q_i,\theta)I^{LS}_{\ell J}(\theta)d(\cos\theta) \quad , \tag{5·16}$$

with

$$\begin{aligned}I^{LS}_{\ell J}(\theta)&\equiv\frac{4\pi}{2J+1}\sum_M\langle\mathcal{Y}_{(\ell\frac{1}{2})JM}(\widehat{\boldsymbol{q}}_f,\mathrm{spin})|i\boldsymbol{n}\boldsymbol{S}_\Lambda|\mathcal{Y}_{(\ell\frac{1}{2})JM}(\widehat{\boldsymbol{q}}_i,\mathrm{spin})\rangle_{\mathrm{spin}}\\&=(-1)^{J+\ell+\frac{1}{2}}\sqrt{\frac{3}{2}}\frac{\widehat{\ell}}{\sqrt{\ell(\ell+1)}}\left\{\begin{array}{ccc}\ell&\frac{1}{2}&J\\\frac{1}{2}&\ell&1\end{array}\right\}q_fq_i\sin\theta P^1_\ell(\cos\theta)\\&=q_fq_i\frac{\left[J(J+1)-\ell(\ell+1)-\frac{3}{4}\right]}{2(2\ell+1)}\left[P_{\ell-1}(\theta)-P_{\ell+1}(\theta)\right] \quad .\end{aligned} \tag{5·17}$$

5.2. *Wigner transform*

For the physical discussion, it is convenient to deal with the local potential in the coordinate representation. It is also possible to discuss the relationship between our nonlocal $\Lambda\alpha$ LS potential and the Scheerbaum factor, where the non-locality is completely swept out by using the short-range character of the LS interaction. This can be done by calculating the Wigner transform of our nonlocal $\Lambda\alpha$ interaction. The transformation between the Wigner transform and the Born kernel is discussed in Appendix B of our Extended Time Reversal paper, Prog. Theor. Phys. 97 (1997) 587 - 616. We calculate the Fourier transform of the Born kernel:

$$\mathcal{O}^W(\boldsymbol{r},\boldsymbol{q})=\frac{1}{(2\pi)^3}\int d\boldsymbol{k}e^{i\boldsymbol{k}\boldsymbol{r}}\mathcal{O}(\boldsymbol{q}_f,\boldsymbol{q}_i) \quad . \tag{5·18}$$

For the direct term, this gives a simple direct potential. The exchange term becomes the momentum-dependent local potential involving $\boldsymbol{q}$, which is called the local momentum.

For the simple Gaussian potential with $u(r) = e^{-\kappa r^2}$, the calculation is very simple and we obtain

$$
\begin{aligned}
V^W_{\Lambda\alpha}(r,q) &= X_d V_d(r) + X_e V^W_e(r,q) \\
V_d(r) &= \left(\frac{1}{1+\frac{3}{8}\frac{\kappa}{\nu}}\right)^{\frac{3}{2}} \exp\left\{-\frac{1}{1+\frac{3}{8}\frac{\kappa}{\nu}}\kappa r^2\right\} \\
V^W_e(r,q) &= \left(\frac{8}{3}\right)^3 \left(\frac{1}{1+\frac{8}{3}\frac{\kappa}{\nu}}\right)^{\frac{3}{2}} \exp\left\{-\frac{8\nu}{3}r^2 - \frac{25}{24}\frac{1}{1+\frac{8}{3}\frac{\kappa}{\nu}}\frac{q^2}{\nu}\right\}
\end{aligned}
$$

$$
\begin{aligned}
V^{LS\ W}_{\Lambda\alpha}(r,q) &= \left[X^{LS}_d V^{LS}_d(r) + X^{LS}_e V^{LS\ W}_e(r,q)\right]\ (\boldsymbol{\ell}\cdot\boldsymbol{S}_\Lambda) \\
V^{LS}_d(r) &= \left(\frac{1}{1+\frac{3}{8}\frac{\kappa}{\nu}}\right)^{\frac{5}{2}} \frac{\zeta+4}{4(\zeta+1)} \exp\left\{-\frac{1}{1+\frac{3}{8}\frac{\kappa}{\nu}}\kappa r^2\right\} \\
V^{LS\ W}_e(r,q) &= \left(\frac{8}{3}\right)^4 \left(\frac{1}{1+\frac{8}{3}\frac{\kappa}{\nu}}\right)^{\frac{5}{2}} \frac{5}{3} \exp\left\{-\frac{8\nu}{3}r^2 - \frac{25}{24}\frac{1}{1+\frac{8}{3}\frac{\kappa}{\nu}}\frac{q^2}{\nu}\right\}\ , \qquad (5{\cdot}19)
\end{aligned}
$$

where we have interpreted $\boldsymbol{L}_W = [\boldsymbol{r}\times\boldsymbol{q}]$ as the orbital angular-momentum operator $\boldsymbol{\ell} = -i[\boldsymbol{r}\times\boldsymbol{\nabla}_r]$ for the $\Lambda\alpha$ relative motion. The transformation of the quark-model LS potential can be carried out in a similar way. We find, for the LS potential without the $(\boldsymbol{\ell}\cdot\boldsymbol{S}_\Lambda)$ factor,

$$
\begin{aligned}
&V^{LS\ W}_{\Lambda\alpha}(r,q) = 2\pi\alpha_S x^3 m_{ud} c^2 b^5 \left(\frac{8\nu}{3\pi}\right)^{\frac{3}{2}} \frac{16\nu}{3} \left\{\frac{\zeta+4}{4(\zeta+1)}\frac{1}{1+\frac{32}{9}b^2\nu}\exp\left\{-\frac{8\nu}{3}\frac{r^2}{1+\frac{32}{9}b^2\nu}\right\}\right. \\
&\times\left[\frac{2}{9\lambda}\left(2+\frac{1}{\lambda}\right)\left(\frac{3}{4}\frac{1}{1+\frac{8}{9}b^2\nu}\right)^{\frac{3}{2}} h_1\left(\left(\frac{8\nu}{3}\right)^2 \frac{(br)^2}{\left(1+\frac{32}{9}b^2\nu\right)\left(1+\frac{8}{9}b^2\nu\right)}\right)\right. \\
&\left.+\frac{2}{9\lambda}\left(2-\frac{1}{\lambda}\right)\left(\frac{12}{11}\frac{1}{1+\frac{8}{9}\frac{32}{11}b^2\nu}\right)^{\frac{3}{2}} h_1\left(\left(\frac{8\nu}{3}\right)\frac{4}{11}\frac{(br)^2}{\left(1+\frac{32}{9}b^2\nu\right)\left(1+\frac{8}{9}\frac{32}{11}b^2\nu\right)}\right)\right] \\
&-\frac{5}{8}\frac{1}{1+\frac{1}{2}b^2\nu}\exp\left\{-\frac{8\nu}{3}r^2\right\}\left[\frac{4}{9\lambda}\left(2+\frac{1}{\lambda}\right)\left(\frac{1}{1+\frac{1}{2}b^2\nu}\right)^{\frac{3}{2}}\exp\left\{-\frac{25}{48}\frac{(bq)^2}{1+\frac{1}{2}b^2\nu}\right\}\right. \\
&\left.\left.+\frac{4}{9\lambda}\left(2-\frac{1}{\lambda}\right)\left(\frac{12}{11+4b^2\nu}\right)^{\frac{3}{2}}\exp\left\{-\frac{25}{6}\frac{(bq)^2}{11+4b^2\nu}\right\}\widetilde{h}_1\left(\frac{5}{4}\frac{bq}{\sqrt{\left(1+\frac{1}{2}b^2\nu\right)(11+4b^2\nu)}}\right)\right]\right\}\ . \qquad (5{\cdot}20)
\end{aligned}
$$

It is interesting to compare the result of Eq. (5·20) with the prediction by the Scheerbaum formalism [R. R. Scheerbaum, Nucl. Phys. A257 (1976) 77]. We first simplify Eq. (5·20) by setting $q = 0$ and $b \to 0$ (point-like baryon) except for the strength factor, $2\pi\alpha_S x^3 m_{ud} c^2 b^5$. If we further set $\zeta = M_\Lambda/M_N \to 1$, the S-type direct and exchange contributions merge together and we get a simple expression

$$
V^{LS\ W}_{\Lambda\alpha}(r,0) \sim -\frac{5}{\sqrt{\pi}}\alpha_S x^3 m_{ud} c^2 \left(b\sqrt{\frac{8\nu}{3}}\right)^5 e^{-\frac{8\nu}{3}r^2}
$$

$$\times \left[\frac{2}{9\lambda}\left(2+\frac{1}{\lambda}\right) - \frac{1}{9\lambda}\left(2+\frac{1}{\lambda}\right)\left(\frac{3}{4}\right)^{\frac{3}{2}} + \frac{1}{9\lambda}\left(2-\frac{1}{\lambda}\right)\left(\frac{12}{11}\right)^{\frac{3}{2}} \right] \quad . \tag{5·21}$$

The last factor of Eq. (5·21) is nothing but the reduced Scheerbaum factor $\widetilde{S}_\Lambda$ for the Λ particle, given in our previous one-body spin-orbit paper, Nucl. Phys. A674 (2000) 493-514. [See Eq. (52).] The first term is the D_- e-type contribution, the second D_+ d-type, and the third the sum of the $S\,d$ and $S\,e$ contributions.

On the other hand, the Scheerbaum formula for the one-body spin-orbit potential is given by

$$U(\boldsymbol{r}) = -\frac{\pi}{2}\; S_B\; \frac{1}{r}\frac{d\rho(r)}{dr}\; \boldsymbol{\ell}\cdot\boldsymbol{\sigma} \quad . \tag{5·22}$$

If we use the Gaussian density distribution of the $(0s)^4$ shell-model wave function for the α-cluster and use the simplification in Eq. (52) [$S_B = -\alpha_S\, x^3\, m_{ud} c^2\, b^5 (\xi/(1+\xi))\widetilde{S}_B$ with $\xi = 1/\zeta$], we find

$$\begin{aligned} U(\boldsymbol{r}) &= S_\Lambda\; \frac{8}{\sqrt{\pi}}(2\nu)^{\frac{5}{2}}\; e^{-2\nu r^2}\; (\boldsymbol{\ell}\cdot\boldsymbol{S}_\Lambda) \\ &= -\frac{8}{\sqrt{\pi}}\frac{1}{\zeta+1}\alpha_S x^3 m_{ud} c^2 \left(b\sqrt{2\nu}\right)^5 \widetilde{S}_\Lambda\; e^{-2\nu r^2}\; (\boldsymbol{\ell}\cdot\boldsymbol{S}_\Lambda) \quad . \end{aligned} \tag{5·23}$$

[Eq. (20) of the above paper includes a misprint. $(4\pi/(2\ell+1))$ should be $((2\ell+1)/4\pi)$.] The reduced Scheerbaum factor $\widetilde{S}_\Lambda$ are given by

$$\begin{aligned} \widetilde{S}_N &= \frac{14}{9} - \frac{10}{27}\left(\frac{3}{4}\right)^{\frac{3}{2}} + \frac{32}{81}\left(\frac{12}{11}\right)^{\frac{3}{2}} \quad , \\ \widetilde{S}_\Lambda &= \frac{2}{9\lambda}\left(2+\frac{1}{\lambda}\right) - \frac{1}{9\lambda}\left(2+\frac{1}{\lambda}\right)\left(\frac{3}{4}\right)^{\frac{3}{2}} + \frac{1}{9\lambda}\left(2-\frac{1}{\lambda}\right)\left(\frac{12}{11}\right)^{\frac{3}{2}} \quad , \\ \widetilde{S}_\Sigma &= \frac{2}{3\cdot 81}\left(106 - \frac{6}{\lambda} - \frac{1}{\lambda^2}\right) - \frac{1}{81}\left(18 - \frac{10}{\lambda} - \frac{3}{\lambda^2}\right)\left(\frac{3}{4}\right)^{\frac{3}{2}} \\ &\quad + \frac{1}{3\cdot 81}\left(26 + \frac{42}{\lambda} - \frac{7}{\lambda^2}\right)\left(\frac{12}{11}\right)^{\frac{3}{2}} \quad , \\ &\quad \vdots \end{aligned} \tag{5·24}$$

If we compare Eq. (5·21) and Eq. (5·23), we find that the difference is only $(8/3)\nu \leftrightarrow 2\nu$ and $5 \leftrightarrow (8/(1+\zeta))$, which may be related to the incorrect treatment of the center-of-mass motion in the Scheerbaum formalism.

5.3. $\boldsymbol{\Lambda\alpha}$ $\boldsymbol{LS}$ *potential in the Scheerbaum formalism*

It is also possible to use the $\Lambda\alpha$ LS potential Eq. (5·23) in the Scheerbaum formalism for the $2\alpha\Lambda$ Faddeev calculation. For this purpose, we calculate the Born kernel of

Eq. (5·23), by using

$$\langle e^{i\boldsymbol{q}_f\boldsymbol{r}}|e^{-\kappa\boldsymbol{r}^2}[\boldsymbol{r}\times\boldsymbol{p}_{\rm op}]|e^{i\boldsymbol{q}_i\boldsymbol{r}}\rangle = \left(\frac{\pi}{\kappa}\right)^{\frac{3}{2}} e^{-\frac{1}{4\kappa}\boldsymbol{k}^2}\frac{1}{2\kappa}\ i\boldsymbol{n}\ , \tag{5·25}$$

with $\boldsymbol{n}=[\boldsymbol{q}\times\boldsymbol{k}]=[\boldsymbol{q}_i\times\boldsymbol{q}_f]$. We find

$$V^{LS}_{\Lambda\alpha}(\boldsymbol{q}_f,\boldsymbol{q}_i)=\langle e^{i\boldsymbol{q}_f\boldsymbol{r}}|U^{LS}_{\Lambda\alpha}(\boldsymbol{r})|e^{i\boldsymbol{q}_i\boldsymbol{r}}\rangle = F(q_f,q_i,\theta)\ i\boldsymbol{n}\cdot\boldsymbol{S}_\Lambda\ , \tag{5·26}$$

with

$$F(q_f,q_i,\theta)=S_\Lambda\ 4\pi\ e^{-\frac{1}{8\nu}\boldsymbol{k}^2}\ , \tag{5·27}$$

with $\boldsymbol{k}=\boldsymbol{q}_f-\boldsymbol{q}_i$. The partial wave component of this $\Lambda\alpha$ potential is calculated from Eqs. (5·15) - (5·17). If we use the relationship

$$i_{\ell-1}(x)-i_{\ell+1}(x)=\frac{2\ell+1}{x}\ i_\ell(x) \tag{5·28}$$

for $i_\ell(x)=i^\ell\ j_\ell(-ix)$, we find

$$V^{LS}_{\ell J}(q_f,q_i)=\frac{1}{2}\left[J(J+1)-\ell(\ell+1)-\frac{3}{4}\right]\ 16\pi\nu\ S_\Lambda\ e^{-\frac{1}{8\nu}\left({q_f}^2+{q_i}^2\right)}\ i_\ell\left(\frac{1}{4\nu}q_fq_i\right)\ . \tag{5·29}$$

5.4. *Scheerbaum factor of the Nijmegen* $\boldsymbol{G}$*-matrix* $\boldsymbol{LS}$ *potentials*

Here, we give a simple expression to calculate the Scheerbaum factor for the Nijmegen G-matrix LS potentials given by Hiyama et al. [Phys. Rev. Lett. 85 (2000) 270] :

$$v^{\Lambda N}_\pm=\left(v^{(1)}_\pm\ e^{-(r/\beta_1)^2}+v^{(2)}_\pm\ e^{-(r/\beta_2)^2}\right)\ \boldsymbol{L}\cdot(\boldsymbol{s}_\Lambda\pm\boldsymbol{s}_N)\ . \tag{5·30}$$

The parameters, $v^{(i)}_\pm$ and β_i $(i=1,\ 2)$ are given in Table I of the same paper. From Eq. (27) of our single particle LS paper [Nucl. Phys. A674 (2000) 493], we find ($\zeta=1/\xi$)

$$S_\Lambda=\frac{1}{2\pi}\frac{1}{1+\zeta}2\sum_{i=1}^{2}\left(v^{(i)}_+ + v^{(i)}_-\right)\ \bar{f}_i\ , \tag{5·31}$$

where the spatial part $\bar{f}_i$ is given by

$$\bar{f}_i=\frac{4\pi}{\bar{q}}\int_0^\infty s^3\ j_1(\bar{q}s)\ e^{-\left(\frac{s}{\beta_i}\right)^2}\ ds\ . \tag{5·32}$$

The integrals are easily calculated by using the formula

$$\begin{aligned}\int_0^\infty x\ \sin x\ e^{-ax^2}dx&=\frac{1}{4a}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}e^{-\frac{1}{4a}}\ ,\\ \int_0^\infty x^2\ \cos x\ e^{-ax^2}dx&=\left(1-\frac{1}{2a}\right)\frac{1}{4a}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}e^{-\frac{1}{4a}}\ .\end{aligned} \tag{5·33}$$

We find

$$\bar{f}_i = \frac{\pi\sqrt{\pi}}{2}\beta^5 \; e^{-\frac{1}{4}(\beta_i\bar{q})^2} \; . \tag{5·34}$$

We finally obtain

$$S_\Lambda = \frac{\sqrt{\pi}}{2}\frac{1}{1+\zeta}\sum_{i=1}^{2}\left(v_+^{(i)} + v_-^{(i)}\right) \; {\beta_i}^5 \; e^{-\frac{1}{4}(\beta_i\bar{q})^2} \; . \tag{5·35}$$

5.5. *σ-meson direct term*

Since the model fss2 also contains the LS component from EMEP, we should also include these contributions in the ℓs splitting calculation of ${}^{9}_{\Lambda}$Be. Here we only include the dominant contribution from the σ- (or ϵ-meson) contribution for simplicity. For the $\Lambda\alpha\alpha$ Faddeev calculation in the jj-coupling scheme, we only need to include

$$F(q_f, q_i, \theta) = -W_{0D_+}^{LS\,\epsilon}(\theta)\frac{2}{3}\left(\frac{1}{m_{ud}}\right)^2 \frac{\zeta+4}{4(\zeta+1)}\frac{4\pi}{\boldsymbol{k}^2 + {m_\epsilon}^2}\exp\left\{-\frac{1}{4\eta_d}\boldsymbol{k}^2\right\} \quad . \tag{5·36}$$

Here m_ϵ is the meson mass and $\eta_d = 1/(3/8\nu + 4b^2/3)$. The Wigner transform of this Born kernel is nothing but the direct potential of the ϵ-meson exchange between quarks.

$$V_{\Lambda\alpha}^{LS\,\epsilon}(r) = -W_{0D_+}^{LS\,\epsilon}(\theta) m_\epsilon \frac{2}{3}\left(\frac{m_\epsilon}{m_{ud}}\right)^2 \frac{\zeta+4}{4(\zeta+1)}\,\widetilde{\mathcal{Z}}^{(1)}_{\frac{{m_\epsilon}^2}{4\eta_d}}(m_\epsilon r) \quad . \tag{5·37}$$

The LS-type Yukawa function for the finite size particles is given by

$$\begin{aligned} Z_\alpha^{(1)}(x) = \frac{e^\alpha}{2x^2}\left[\left(1+\frac{1}{x}\right)e^{-x}\mathrm{erfc}\left(\sqrt{\alpha}-\frac{x}{2\sqrt{\alpha}}\right) + \left(1-\frac{1}{x}\right)e^{x}\mathrm{erfc}\left(\sqrt{\alpha}+\frac{x}{2\sqrt{\alpha}}\right)\right] \\ -\frac{1}{\sqrt{\alpha\pi}}\frac{1}{x^2}\exp\left\{-\frac{x^2}{4\alpha}\right\} \quad . \end{aligned} \tag{5·38}$$

In the above equations, $W_{0D_+}^{LS\epsilon}(\theta)$ is essentially the scalar-meson coupling constants, including the singlet-octet mixing, and is explicitly given by

$$\begin{aligned} W_{0D_+}^{LS\epsilon}(\theta) = \frac{1}{2}\left[f_1\cos\theta\left(\frac{1}{3}f_1\cos\theta - \frac{1}{\sqrt{3}}f_8\sin\theta\right)\right. \\ \left. + \left(f_1\cos\theta - \sqrt{3}f_8\sin\theta\right)\left(\frac{1}{3}f_1\cos\theta + \frac{2}{\sqrt{3}}f_8\sin\theta\right)\right] \quad . \end{aligned} \tag{5·39}$$

Here f_1, f_8 and θ are scalar-meson SU_3 coupling constants, which are given in our fss2 paper as

$$f_1 = 3.48002 \quad , \qquad f_8 = 0.94459 \quad , \qquad \theta = 33.3295^\circ \quad . \tag{5·40}$$

5.6. *$\Lambda\alpha$ interaction using the Born kernel*

We can also calculate the $\Lambda\alpha$ kernel and its Wigner transform, by directly using the ΛN Born kernel, Eqs. (9) and (A.3), given in our s.p. LS paper (Nucl. Phys. A674 (2000) 493 - 514). The easiest way to calculate these is to start from the original expression

$$f_{\mathcal{T}}^{LS}(\theta) = (-2\pi)\alpha_S x^3 m_{ud} c^2 b^5 \, e^{-\frac{1}{3}b^2(\boldsymbol{q}^2+\boldsymbol{k}^2)} \times \begin{cases} h_1(-\boldsymbol{\rho}^2) & \text{with} \quad \boldsymbol{\rho} = \frac{1}{\sqrt{3}} b\boldsymbol{q} & D_- \\ \left(\frac{3}{4}\right)^{\frac{3}{2}} h_1(-\boldsymbol{\rho}^2) & \text{with} \quad \boldsymbol{\rho} = \frac{1}{2} b\boldsymbol{k} & D_+ \\ \left(\frac{12}{11}\right)^{\frac{3}{2}} h_1(-\boldsymbol{\rho}^2) & \text{with} \quad \boldsymbol{\rho} = \frac{1}{\sqrt{11}} b(\boldsymbol{q}+\boldsymbol{k}) & S \end{cases} , \tag{5·41}$$

or

$$f_{\mathcal{T}}^{LS}(\theta) = (-2\pi)\alpha_S x^3 m_{ud} c^2 b^5 3 \int_0^1 t^2 dt \, e^{-\frac{1}{3}b^2(\boldsymbol{q}^2+\boldsymbol{k}^2)} \times \begin{cases} e^{\frac{t^2}{3}(b\boldsymbol{q})^2} & D_- \\ \left(\frac{3}{4}\right)^{\frac{3}{2}} e^{\frac{t^2}{4}(b\boldsymbol{k})^2} & D_+ \\ \left(\frac{12}{11}\right)^{\frac{3}{2}} e^{\frac{t^2}{11}b^2(\boldsymbol{q}+\boldsymbol{k})^2} & S \end{cases} . \tag{5·42}$$

We first rewrite Eq. (5·4) as

$$V(\boldsymbol{q}_f, \boldsymbol{q}_i) = e^{-\frac{3}{4}\frac{\boldsymbol{k}^2}{8\nu}} \left(\frac{b^2}{3\pi\varepsilon}\right)^{\frac{3}{2}} \int d\boldsymbol{q}' e^{-\frac{b^2}{3\varepsilon}(\boldsymbol{q}'-\boldsymbol{q})^2} u\left(\boldsymbol{k}, \boldsymbol{q}'\right) ,$$
$$\text{with} \qquad \varepsilon = \frac{1}{2}\left(\frac{\zeta}{\zeta+1}\right)^2 \nu b^2 \quad , \qquad \widetilde{\boldsymbol{q}} = \frac{\zeta+4}{4(\zeta+1)}\boldsymbol{q} \, . \tag{5·43}$$

Here we have used the fact that the quark-model Born kernel $u(\boldsymbol{k}', \boldsymbol{q}', \boldsymbol{P})$ is independent of $\boldsymbol{P}$. If we apply this formula to $u(\boldsymbol{k}', \boldsymbol{q}') = \exp\{-b^2(\alpha \boldsymbol{k}'^2 + \beta \boldsymbol{q}'^2 + \gamma \boldsymbol{k}' \cdot \boldsymbol{q}')\}$, we obtain the $\Lambda\alpha$ Born kernel

$$V^{\left\{\begin{smallmatrix} C \\ LS \end{smallmatrix}\right\}}(\boldsymbol{q}_f, \boldsymbol{q}_i) = \begin{Bmatrix} 1 \\ \frac{\zeta+4}{4(\zeta+1)} \frac{1}{1+3\varepsilon\beta} \end{Bmatrix} \left(\frac{1}{1+3\varepsilon\beta}\right)^{\frac{3}{2}} e^{-\frac{3}{4}\frac{\boldsymbol{k}^2}{8\nu}}$$
$$\times \exp\left\{-\left(\alpha - \frac{3\varepsilon}{4(1+3\varepsilon\beta)}\gamma^2\right)(b\boldsymbol{k})^2 - \frac{1}{1+3\varepsilon\beta} b^2(\beta\widetilde{\boldsymbol{q}}^2 + \gamma \boldsymbol{k}\cdot\widetilde{\boldsymbol{q}})\right\} , \tag{5·44}$$

and the Wigner transform

$$G_W^{\left\{\begin{smallmatrix} C \\ LS \end{smallmatrix}\right\}}(\boldsymbol{r}, \boldsymbol{q}) = \begin{Bmatrix} 1 \\ \frac{\zeta+4}{4(\zeta+1)} \frac{2\eta}{1+3\varepsilon\beta} \end{Bmatrix} \left(\frac{1}{\pi}\frac{\eta}{1+3\varepsilon\beta}\right)^{\frac{3}{2}} \exp\left\{-\eta\boldsymbol{r}^2 + i2\frac{\eta b^2}{1+3\varepsilon\beta}\gamma\boldsymbol{r}\cdot\widetilde{\boldsymbol{q}}\right.$$
$$\left. -\frac{\eta}{1+3\varepsilon\beta}\left[\frac{3}{8\nu}\beta + (4\alpha\beta-\gamma^2)b^2\right](b\widetilde{\boldsymbol{q}})^2\right\} , \tag{5·45}$$

where

$$\eta = \frac{1}{\frac{3}{8\nu} + \frac{1}{1+3\varepsilon\beta}[4\alpha + 3\varepsilon(4\alpha\beta-\gamma^2)]b^2} \, . \tag{5·46}$$

For the direct term, we apply this formula to

$$u(\boldsymbol{k},\boldsymbol{q}) = e^{-\frac{1}{3}b^2(\boldsymbol{q}^2+\boldsymbol{k}^2)+b^2(\sqrt{\beta}\boldsymbol{q}+\sqrt{\alpha}\boldsymbol{k})^2t^2} \ . \tag{5·47}$$

For the exchange term, we make a formula for

$$u(\boldsymbol{k},\boldsymbol{q}) = e^{-\frac{1}{3}b^2(\frac{1}{4}\boldsymbol{k}^2+4\boldsymbol{q}^2)+b^2(\frac{1}{2}\sqrt{\beta}\boldsymbol{k}+2\sqrt{\alpha}\boldsymbol{q})^2t^2} \ . \tag{5·48}$$

The t integration is carried out by the formula

$$\begin{aligned} &3\int_0^1 t^2dt\left(\frac{1}{A-Bt^2}\right)^{\frac{5}{2}} e^{-\frac{C-Dt^2}{A-Bt^2}} \\ &= \begin{cases} \left(\frac{1}{A-B}\right)^{\frac{3}{2}}\frac{1}{A}e^{-\frac{C-D}{A-B}}\,\widetilde{h}_1\left(\sqrt{\frac{AD-BC}{A(A-B)}}\right) & \text{for} \quad AD-BC>0\ , \\ \left(\frac{1}{A-B}\right)^{\frac{3}{2}}\frac{1}{A}e^{-\frac{C}{A}}\,h_1\left(\frac{BC-AD}{A(A-B)}\right) & \text{for} \quad BC-AD>0\ . \end{cases} \end{aligned} \tag{5·49}$$

The upper case is applied to the Born kernel and the lower to the Wigner transform, in general. From this procedure, we find, for the direct term

$$\begin{aligned} &V_d^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i) = \frac{\zeta+4}{4(\zeta+1)}e^{-\frac{3}{4}\frac{\boldsymbol{k}^2}{8\nu}}\left(\frac{1}{1+(1-3\beta)\varepsilon}\right)^{\frac{3}{2}}\frac{1}{1+\varepsilon} \\ &\times\exp\left\{-\frac{b^2}{1+(1-3\beta)\varepsilon}\left[\left(\left(\frac{1}{3}-\alpha\right)+\left(\frac{1}{3}-\alpha-\beta\right)\varepsilon\right)\boldsymbol{k}^2+\left(\frac{1}{3}-\beta\right)\widetilde{\boldsymbol{q}}-2\sqrt{\alpha\beta}\boldsymbol{k}\cdot\widetilde{\boldsymbol{q}}\right]\right\} \\ &\times\widetilde{h}_1\left(\frac{b|\sqrt{\alpha}(1+\varepsilon)\boldsymbol{k}+\sqrt{\beta}\widetilde{\boldsymbol{q}}|}{\sqrt{(1+\varepsilon)(1+(1-3\beta)\varepsilon)}}\right)\ , \end{aligned}$$

$$\begin{aligned} &G_{dW}^{LS}(\boldsymbol{r},\boldsymbol{q}) = \frac{\zeta+4}{2(\zeta+1)}\frac{1}{(1+\varepsilon)\left(\frac{3}{8\nu}+\frac{4}{3}b^2\right)}\left(\frac{1}{\pi}\frac{1}{D}\right)^{\frac{3}{2}}\exp\left\{-\frac{\boldsymbol{r}^2}{\frac{3}{8\nu}+\frac{4}{3}b^2}-\frac{1}{1+\varepsilon}\frac{1}{3}(b\widetilde{\boldsymbol{q}})^2\right\} \\ &\times h_1\left(\frac{b^2}{D}\left(2\sqrt{\alpha}\sqrt{\frac{1+\varepsilon}{\frac{3}{8\nu}+\frac{4}{3}b^2}}\boldsymbol{r}+i\sqrt{\beta}\sqrt{\frac{\frac{3}{8\nu}+\frac{4}{3}b^2}{1+\varepsilon}}\widetilde{\boldsymbol{q}}\right)^2\right)\ , \\ &\qquad\text{with}\quad D=\frac{3}{8\nu}(1+(1-3\beta)\varepsilon)+4b^2\left(\left(\frac{1}{3}-\alpha\right)+\left(\frac{1}{3}-\alpha-\beta\right)\varepsilon\right)\ . \end{aligned} \tag{5·50}$$

The formula for the exchange term is

$$\begin{aligned} &V_e^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i) = \frac{\zeta+4}{4(\zeta+1)}e^{-\frac{3}{4}\frac{\boldsymbol{k}^2}{8\nu}}\left(\frac{1}{1+4(1-3\alpha)\varepsilon}\right)^{\frac{3}{2}}\frac{1}{1+4\varepsilon}\exp\left\{-\frac{b^2}{1+4(1-3\alpha)\varepsilon}\right. \\ &\left.\times\left[\left(\frac{1}{4}\left(\frac{1}{3}-\beta\right)+\left(\frac{1}{3}-\alpha-\beta\right)\varepsilon\right)\boldsymbol{k}^2+4\left(\frac{1}{3}-\alpha\right)\widetilde{\boldsymbol{q}}-2\sqrt{\alpha\beta}\boldsymbol{k}\cdot\widetilde{\boldsymbol{q}}\right]\right\} \\ &\times\widetilde{h}_1\left(\frac{b|\frac{1}{2}\sqrt{\beta}(1+4\varepsilon)\boldsymbol{k}+2\sqrt{\alpha}\widetilde{\boldsymbol{q}}|}{\sqrt{(1+4\varepsilon)(1+4(1-3\alpha)\varepsilon)}}\right)\ , \end{aligned}$$

$$G_{eW}^{LS}(\boldsymbol{r},\boldsymbol{q}) = \frac{\zeta+4}{2(\zeta+1)}\frac{1}{(1+4\varepsilon)\left(\frac{3}{8\nu}+\frac{b^2}{3}\right)}\left(\frac{1}{\pi}\frac{1}{D}\right)^{\frac{3}{2}}\exp\left\{-\frac{\boldsymbol{r}^2}{\frac{3}{8\nu}+\frac{b^2}{3}}-\frac{1}{1+4\varepsilon}\frac{4}{3}(b\widetilde{\boldsymbol{q}})^2\right\}$$

$$\times h_1\left(\frac{b^2}{D}\left(\sqrt{\beta}\sqrt{\frac{1+4\varepsilon}{\frac{3}{8\nu}+\frac{b^2}{3}}}\boldsymbol{r}+i2\sqrt{\alpha}\sqrt{\frac{\frac{3}{8\nu}+\frac{b^2}{3}}{1+4\varepsilon}}\widetilde{\boldsymbol{q}}\right)^2\right)\ ,$$

$$\text{with}\quad D=\frac{3}{8\nu}(1+4(1-3\alpha)\varepsilon)+b^2\left(\left(\frac{1}{3}-\beta\right)+4\left(\frac{1}{3}-\alpha-\beta\right)\varepsilon\right)\ . \tag{5·51}$$

From these formulas, we obtain the $\Lambda\alpha$ Born kernel as

$$V^{LS}_{\mathcal{T}\,d}(\boldsymbol{q}_f,\boldsymbol{q}_i)=(-2\pi)\alpha_S x^3 m_{ud}c^2b^5\frac{\zeta+4}{4(\zeta+1)}\frac{1}{1+\varepsilon}e^{-\frac{3}{4}\frac{\boldsymbol{k}^2}{8\nu}}$$
$$\times\begin{cases} e^{-\frac{1}{3}(b\boldsymbol{k})^2}\,\widetilde{h}_1\left(\frac{1}{\sqrt{1+\varepsilon}}\frac{1}{\sqrt{3}}b|\widetilde{\boldsymbol{q}}|\right) & D_-\,d\\ \left(\frac{3}{4}\frac{1}{1+\varepsilon}\right)^{\frac{3}{2}}e^{-\frac{b^2}{3}\left(\frac{1}{4}\boldsymbol{k}^2+\frac{1}{1+\varepsilon}\widetilde{\boldsymbol{q}}^2\right)}\,\widetilde{h}_1\left(\frac{1}{2}b|\boldsymbol{k}|\right) & D_+\,d\\ \left(\frac{12}{11+8\varepsilon}\right)^{\frac{3}{2}}\exp\left\{-\frac{b^2}{11+8\varepsilon}\left[\left(\frac{8}{3}+\frac{5}{3}\varepsilon\right)\boldsymbol{k}^2+\frac{8}{3}\widetilde{\boldsymbol{q}}^2-2\boldsymbol{k}\cdot\widetilde{\boldsymbol{q}}\right]\right\}\,\widetilde{h}_1\left(\frac{b|(1+\varepsilon)\boldsymbol{k}+\widetilde{\boldsymbol{q}}|}{\sqrt{(1+\varepsilon)(11+8\varepsilon)}}\right) & S\,d\end{cases}$$

$$V^{LS}_{\mathcal{T}\,e}(\boldsymbol{q}_f,\boldsymbol{q}_i)=(-2\pi)\alpha_S x^3 m_{ud}c^2b^5\frac{\zeta+4}{4(\zeta+1)}\frac{1}{1+4\varepsilon}e^{-\frac{3}{4}\frac{\boldsymbol{k}^2}{8\nu}}$$
$$\times\begin{cases} \left(\frac{1}{1+4\varepsilon}\right)^{\frac{3}{2}}e^{-\frac{1}{1+4\varepsilon}\frac{4}{3}(b\widetilde{\boldsymbol{q}})^2}\,\widetilde{h}_1\left(\frac{1}{2\sqrt{3}}b|\widetilde{\boldsymbol{k}}|\right) & D_-\,e\\ \left(\frac{3}{4}\frac{1}{1+\varepsilon}\right)^{\frac{3}{2}}e^{-\frac{b^2}{3}\left(\frac{1}{4}\boldsymbol{k}^2+\frac{1}{1+\varepsilon}\widetilde{\boldsymbol{q}}^2\right)}\,\widetilde{h}_1\left(\frac{b|\widetilde{\boldsymbol{q}}|}{\sqrt{(1+\varepsilon)(1+4\varepsilon)}}\right) & D_+\,e\\ \left(\frac{12}{11+32\varepsilon}\right)^{\frac{3}{2}}\exp\left\{-\frac{b^2}{11+32\varepsilon}\left[\left(\frac{2}{3}+\frac{5}{3}\varepsilon\right)\boldsymbol{k}^2+\frac{32}{3}\widetilde{\boldsymbol{q}}^2-2\boldsymbol{k}\cdot\widetilde{\boldsymbol{q}}\right]\right\}\,\widetilde{h}_1\left(\frac{b|\frac{1}{2}(1+4\varepsilon)\boldsymbol{k}+2\widetilde{\boldsymbol{q}}|}{\sqrt{(1+4\varepsilon)(11+32\varepsilon)}}\right) & S\,e\end{cases} \tag{5·52}$$

The S'-type kernel is obtained from the S-type kernel by $\boldsymbol{k}\to-\boldsymbol{k}$. The Wigner transform is given by

$$G^{LS}_{\mathcal{T}\,d\,W}(\boldsymbol{r},\boldsymbol{q})=(-2\pi)\alpha_S x^3 m_{ud}c^2b^5\frac{\zeta+4}{2(\zeta+1)}\frac{\eta_d}{1+\varepsilon}\left(\frac{\eta_d}{\pi}\right)^{\frac{3}{2}}e^{-\eta_d\boldsymbol{r}^2}$$
$$\times\begin{cases} \widetilde{h}_1\left(\frac{1}{\sqrt{1+\varepsilon}}\frac{1}{\sqrt{3}}b|\widetilde{\boldsymbol{q}}|\right) & D_-\,d\\ \left(\frac{3}{4}\frac{1}{1+\varepsilon}\frac{1+\frac{32}{9}\nu b^2}{1+\frac{8}{9}\nu b^2}\right)^{\frac{3}{2}}e^{-\frac{1}{1+\varepsilon}\frac{1}{3}(b\widetilde{\boldsymbol{q}})^2}\,h_1\left(\frac{8}{3}\nu b^2\frac{\eta_d\boldsymbol{r}^2}{1+\frac{8}{9}\nu b^2}\right) & D_+\,d\\ \left(\frac{12\left(1+\frac{32}{9}\nu b^2\right)}{11+8\varepsilon+\frac{32}{9}\nu b^2(8+5\varepsilon)}\right)^{\frac{3}{2}}e^{-\frac{1}{1+\varepsilon}\frac{1}{3}(b\widetilde{\boldsymbol{q}})^2}\,h_1\left(\frac{\frac{8}{3}\nu b^2\left(2\sqrt{\eta_d(1+\varepsilon)}\boldsymbol{r}+i\frac{1}{\sqrt{\eta_d(1+\varepsilon)}}\widetilde{\boldsymbol{q}}\right)^2}{11+8\varepsilon+\frac{32}{9}\nu b^2(8+5\varepsilon)}\right) & S\,d\end{cases}$$

$$G^{LS}_{\mathcal{T}\,e\,W}(\boldsymbol{r},\boldsymbol{q})=(-2\pi)\alpha_S x^3 m_{ud}c^2b^5\frac{\zeta+4}{2(\zeta+1)}\frac{\eta_e}{1+4\varepsilon}\left(\frac{\eta_e}{\pi}\right)^{\frac{3}{2}}e^{-\eta_e\boldsymbol{r}^2}$$
$$\times\begin{cases} \left(\frac{1+\frac{8}{9}\nu b^2}{1+4\varepsilon}\right)^{\frac{3}{2}}e^{-\frac{1}{1+4\varepsilon}\frac{4}{3}(b\widetilde{\boldsymbol{q}})^2}\,h_1\left(\frac{8}{9}\nu b^2\eta_e\boldsymbol{r}^2\right) & D_-\,e\\ \left(\frac{3}{4}\frac{1}{1+\varepsilon}\right)^{\frac{3}{2}}e^{-\frac{1}{1+\varepsilon}\frac{1}{3}(b\widetilde{\boldsymbol{q}})^2}\,\widetilde{h}_1\left(\frac{b|\widetilde{\boldsymbol{q}}|}{\sqrt{(1+\varepsilon)(1+4\varepsilon)}}\right) & D_+\,e\\ \left(\frac{12\left(1+\frac{8}{9}\nu b^2\right)}{11+32\varepsilon+\frac{32}{9}\nu b^2(2+5\varepsilon)}\right)^{\frac{3}{2}}e^{-\frac{1}{1+4\varepsilon}\frac{4}{3}(b\widetilde{\boldsymbol{q}})^2}\,h_1\left(\frac{\frac{8}{3}\nu b^2\left(\sqrt{\eta_e(1+4\varepsilon)}\boldsymbol{r}+i\frac{2}{\sqrt{\eta_e(1+4\varepsilon)}}\widetilde{\boldsymbol{q}}\right)^2}{11+32\varepsilon+\frac{32}{9}\nu b^2(2+5\varepsilon)}\right) & S\,d\end{cases} \tag{5·53}$$

Here, we have used

$$\eta_d = \frac{1}{\frac{3}{8\nu} + \frac{4}{3}b^2} \quad , \qquad \eta_e = \frac{1}{\frac{3}{8\nu} + \frac{b^2}{3}} \ . \tag{5·54}$$

The S'-type Wigner transform is obtained from the S-type one by taking the complex conjugate.

The transition to the Scheerbaum potential is very easy if we use the Wigner transform Eq. (5·53). We only need to take $b \to 0$, except for the strength factor $(-2\pi)\alpha_S x^3 m_{ud} c^2 b^5$, and change, $\eta_d = \eta_e = (8\nu/3) \to 2\nu$. Then we find

$$G^{LS}_{\Lambda\alpha\ W}(\boldsymbol{r}, \boldsymbol{q}) \to S^{\rm eff}_\Lambda \frac{8}{\sqrt{\pi}} (2\nu)^{\frac{5}{2}}\, e^{-2\nu \boldsymbol{r}^2} \ , \tag{5·55}$$

with

$$S^{\rm eff}_\Lambda = (-2\pi)\alpha_S x^3 m_{ud} c^2 b^5 \frac{\zeta+4}{4(\zeta+1)} \widetilde{S}_\Lambda = \frac{\zeta+4}{4}\, S_\Lambda \ , \tag{5·56}$$

where $\widetilde{S}_\Lambda$ is given by Eq. (5·24).

The transition from Eq. (5·52) to Eq. (5·14) is achieved in the following way. The direct term in Eq. (5·14) is obtained by setting $\boldsymbol{q} = 0$ and $\varepsilon = 0$ in Eq. (5·52). The exchange term involves a little complication, since the different treatment of the relative momentum between in the local potential and in the quark-model Born kernel. In the former case, the correct expression $\boldsymbol{p} = (\boldsymbol{p}_1 - \zeta \boldsymbol{p}_2)/(\zeta+1)$ between Λ and N is used, while in the latter case a simple expression $\boldsymbol{p} = (\boldsymbol{p}_1 - \boldsymbol{p}_2)/2$ is used. Therefore, the precise agreement between Eq. (5·52) and Eq. (5·14) is obtained only when we further use the approximation $\zeta \to 1$. The same situation takes place in the correspondence of the Wigner transform, Eq. (5·53) and Eq. (5·20).

5.7. *$\Lambda\alpha$ Born kernel for EMEP*

We start from the LS kernel of the EMEP, given in the Appendix A of our fss2 paper (Phys. Rev. C65 (2002) 014002):

$$M^{LS\binom{S}{V}}(\boldsymbol{q}_f, \boldsymbol{q}_i) = -\begin{pmatrix} g^2(b\gamma)^2 \\ f_m f_e 4b^2\gamma \end{pmatrix} \left[X^{LS}_{0D_+} f^{LS}_D(\theta) + X^{LS}_{1D_+} f^{LS}_{D_+}(\theta) - X^{LS}_{1D_-} f^{LS}_{D_-}(\theta) \right] \ , \tag{5·57}$$

where $\gamma = (m/2m_{ud})$ and the spatial integrals are given by (see Eqs. (A9) and (A10) of the above paper)

$$\begin{aligned} f^{LS}_D(\theta) &= \hbar c \frac{1}{3\alpha_E} \frac{4\pi}{\boldsymbol{k}^2 + m^2} e^{-\frac{1}{3}(b\boldsymbol{k}^2)} \ , \\ f^{LS}_{\binom{D_-}{D_+}}(\theta) &= 6\pi\hbar c b^2 \begin{cases} e^{-\frac{1}{3}(b\boldsymbol{k})^2}\, \widetilde{\mathcal{Z}}^{(1)}_{\alpha_{D_-}}\left(\frac{1}{\sqrt{3}} b|\boldsymbol{q}|\right) \ , \\ \left(\frac{3}{4}\right)^{\frac{1}{2}} e^{-\frac{1}{3}\left(\boldsymbol{q}^2 + \frac{1}{4}\boldsymbol{k}^2\right)} \widetilde{\mathcal{Z}}^{(1)}_{\alpha_{D_+}}\left(\frac{1}{2} b|\boldsymbol{k}|\right) \ . \end{cases} \end{aligned} \tag{5·58}$$

Here, $\alpha_E = (1/2)(mb)^2$, $\alpha_{D_-} = (3/2)\alpha_E$, and $\alpha_{D_+} = 2\alpha_E$. For the direct term with no $\boldsymbol{q}$-dependence, we only need to multiply the factor

$$\frac{\zeta+4}{4(\zeta+1)} e^{-\frac{3}{4}\frac{\boldsymbol{k}^2}{8\nu}} \ . \tag{5·59}$$

For the one-quark exchange term, we use the definition

$$\widetilde{\mathcal{Z}}_\alpha^{(1)}(\rho) = \frac{1}{2\alpha}\int_0^1 t^2 dt\ e^{\alpha\left(1-\frac{1}{t^2}\right)+\rho^2(1-t^2)} \ , \tag{5·60}$$

and write Eq. (5·58) as

$$f^{LS}_{\binom{D_-}{D_+}}(\theta) = 6\pi\hbar c b^2 \frac{1}{2\alpha}\int_0^1 t^2 dt\ e^{\alpha\left(1-\frac{1}{t^2}\right)} e^{-\frac{b^2}{3}(\boldsymbol{q}^2+\boldsymbol{k}^2)} \begin{cases} e^{\frac{t^2}{3}(b\boldsymbol{q})^2} \ , \\ \left(\frac{3}{4}\right)^{\frac{1}{2}} e^{\frac{t^2}{4}(\boldsymbol{k}^2)} \ . \end{cases} \tag{5·61}$$

The transformation formulas in Eq. (5·44) and Eq. (5·45) are simplified, if we set $\gamma = 0$:

$$\begin{aligned} V^{LS}(\boldsymbol{q}_f, \boldsymbol{q}_i) &= \frac{\zeta+4}{4(\zeta+1)} e^{-\frac{3}{4}\frac{\boldsymbol{k}^2}{8\nu}} \left(\frac{1}{1+3\varepsilon\beta}\right)^{\frac{5}{2}} e^{-\alpha(b\boldsymbol{k})^2 - \frac{\beta}{1+3\varepsilon\beta}(b\boldsymbol{q})^2} \ , \\ G^{LS}_W(\boldsymbol{r}, \boldsymbol{q}) &= \frac{\zeta+4}{2(\zeta+1)} \left(\frac{1}{\pi}\right)^{\frac{3}{2}} \left(\frac{\eta}{1+3\varepsilon\beta}\right)^{\frac{5}{2}} e^{-\eta\boldsymbol{r}^2 - \frac{\beta}{1+3\varepsilon\beta}(b\boldsymbol{q})^2} \ , \end{aligned} \tag{5·62}$$

with $\eta = 1/(3/8\nu + 4\alpha b^2)$, for $u(\boldsymbol{k}, \boldsymbol{q}) = e^{-b^2(\alpha\boldsymbol{k}^2+\beta\boldsymbol{q}^2)}$. The exchange terms are also transformed by these formulas after modifying $\boldsymbol{q} \to \boldsymbol{k}/2$ and $\boldsymbol{k} \to 2\boldsymbol{q}$ in Eq. (5·61). The final integrals over t are reduced to the function Eq. (5·60) or

$$Z_\alpha^{(1)}(\rho) = \frac{1}{2\alpha\sqrt{\pi\alpha}}\int_0^1 t^2 dt\ e^{\alpha\left(1-\frac{1}{t^2}\right)-\frac{\rho^2}{4\alpha}t^2} \ , \tag{5·63}$$

by using the formulas

$$\begin{aligned} &\frac{1}{2\alpha}\int_0^1 t^2 dt\ e^{\alpha\left(1-\frac{1}{t^2}\right)} \left(\frac{1}{a-bt^2}\right)^{\frac{5}{2}} e^{-\frac{c-dt^2}{a-bt^2}} = \frac{1}{a^2\sqrt{a-b}} e^{-\frac{c-d}{a-b}} \widetilde{\mathcal{Z}}^{(1)}_{\frac{a-b}{a}\alpha}\left(\sqrt{\frac{ad-bc}{a(a-b)}}\right) \\ &\qquad \text{for} \quad ad-bc>0 \quad \text{and} \quad a>b \ , \\ &\frac{1}{2\alpha}\int_0^1 t^2 dt\ e^{\alpha\left(1-\frac{1}{t^2}\right)} \left(\frac{1}{a-bt^2}\right)^{\frac{5}{2}} e^{-\frac{\boldsymbol{r}^2}{a-bt^2}} = \left(\frac{1}{a}\right)^{\frac{5}{2}} e^{-\frac{\boldsymbol{r}^2}{a}} \sqrt{\pi\alpha} Z^{(1)}_{\frac{a-b}{a}\alpha}\left(\frac{2\sqrt{b\alpha}}{a}|\boldsymbol{r}|\right) \end{aligned} \tag{5·64}$$

For the transformation of the $\Lambda\alpha$ exchange term for $f_D^{LS}(\theta)$, we use the formula Eq. (A.16) in our LS-RGM paper:

$$\frac{4\pi}{\boldsymbol{k}^2+m^2} = m\frac{1}{\sqrt{\pi}}\int_0^\infty du\ e^{-\frac{1}{u^2}} \left(\frac{\pi}{\kappa}\right)^{\frac{3}{2}} e^{-\frac{\boldsymbol{k}^2}{4\kappa}} \qquad \text{with} \quad \kappa = \left(\frac{mu}{2}\right)^2 \ . \tag{5·65}$$

We find

$$\frac{1}{(b\boldsymbol{k})^2+2\alpha} e^{-\frac{1}{3}(b\boldsymbol{k})^2} = \frac{1}{\alpha}\int_0^\infty du\ \frac{1}{u^3}\ e^{-\frac{1}{u^2}-\left(\frac{1}{3}+\frac{1}{2\alpha u^2}\right)(b\boldsymbol{k})^2} \ . \tag{5·66}$$

We then change $\boldsymbol{k} \to 2\boldsymbol{q}$ and apply the formula Eq. (5·62). The restoration of the u-integral is carried out by using another formula:

$$\int_0^\infty u^2 du \left(\frac{1}{1+\alpha u^2}\right)^{\frac{5}{2}} e^{-\frac{1}{u^2}-\frac{\rho^2}{1+\alpha u^2}} = \frac{2}{\sqrt{\alpha}} \widetilde{\mathcal{Z}}^{(1)}_{\alpha}(\rho) \tag{5·67}$$

The similar integral representation for $Z^{(1)}_{\alpha}(\rho)$ is

$$Z^{(1)}_{\alpha}(\rho) = \frac{1}{2\sqrt{\pi}} \int_0^\infty u^2 du \left(\frac{1}{1+\alpha u^2}\right)^{\frac{5}{2}} e^{-\frac{1}{u^2}-\frac{u^2}{1+\alpha u^2}\frac{\rho^2}{4}} \ . \tag{5·68}$$

We can use this representation, in order to derive the Wigner transform for the $\Lambda\alpha$ direct term for $f^{LS}_D(\theta)$. From these procedures, we can derive the following results for the spatial integrals for EMEP's, which have very similar structure to Eq. (5·52) and Eq. (5·53):

$$V^{LS}_{D\,d}(\boldsymbol{q}_f, \boldsymbol{q}_i) = \hbar c \frac{\zeta+4}{4(\zeta+1)} e^{-\frac{3}{4}\frac{\boldsymbol{k}^2}{8\nu}} \frac{1}{3\alpha_E} \frac{4\pi}{\boldsymbol{k}^2+m^2} e^{-\frac{1}{3}(b\boldsymbol{k}^2)} \ ,$$

$$V^{LS}_{\binom{D_-}{D_+}d}(\boldsymbol{q}_f, \boldsymbol{q}_i) = 6\pi\hbar c b^2 \frac{\zeta+4}{4(\zeta+1)} \left(\frac{1}{1+\varepsilon}\right)^2 e^{-\frac{3}{4}\frac{\boldsymbol{k}^2}{8\nu}}$$
$$\times \begin{cases} e^{-\frac{1}{3}(b\boldsymbol{k})^2} \widetilde{\mathcal{Z}}^{(1)}_{\frac{1}{1+\varepsilon}\alpha_{D_-}} \left(\frac{1}{\sqrt{1+\varepsilon}} \frac{1}{\sqrt{3}} b|\widetilde{\boldsymbol{q}}|\right) \\ \left(\frac{3}{4}\frac{1}{1+\varepsilon}\right)^{\frac{1}{2}} e^{-\frac{b^2}{3}\left(\frac{1}{4}\boldsymbol{k}^2+\frac{1}{1+\varepsilon}\widetilde{\boldsymbol{q}}^2\right)} \widetilde{\mathcal{Z}}^{(1)}_{\alpha_{D_+}} \left(\frac{1}{2} b|\boldsymbol{k}|\right) \end{cases} \ ,$$

$$V^{LS}_{D\,e}(\boldsymbol{q}_f, \boldsymbol{q}_i) = \hbar c b^2 \frac{\zeta+4}{4(\zeta+1)} e^{-\frac{3}{4}\frac{\boldsymbol{k}^2}{8\nu}} \frac{2\pi}{3} \left(\frac{1}{3\varepsilon}\right)^2 \frac{1}{\sqrt{1+4\varepsilon}} e^{-\frac{1}{4\varepsilon}\frac{4}{3}(b\widetilde{\boldsymbol{q}})^2} \widetilde{\mathcal{Z}}^{(1)}_{\frac{1+4\varepsilon}{6\varepsilon}\alpha_E} \left(\frac{b|\widetilde{\boldsymbol{q}}|}{\sqrt{3\varepsilon(1+4\varepsilon)}}\right) \ ,$$

$$V^{LS}_{\binom{D_-}{D_+}e}(\boldsymbol{q}_f, \boldsymbol{q}_i) = 6\pi\hbar c b^2 \frac{\zeta+4}{4(\zeta+1)} \left(\frac{1}{1+4\varepsilon}\right)^2 e^{-\frac{3}{4}\frac{\boldsymbol{k}^2}{8\nu}}$$
$$\times \begin{cases} \left(\frac{1}{1+4\varepsilon}\right)^{\frac{1}{2}} e^{-\frac{1}{1+4\varepsilon}\frac{4}{3}(b\widetilde{\boldsymbol{q}})^2} \widetilde{\mathcal{Z}}^{(1)}_{\alpha_{D_-}} \left(\frac{1}{2\sqrt{3}} b|\widetilde{\boldsymbol{k}}|\right) \\ \left(\frac{3}{4}\frac{1}{1+\varepsilon}\right)^{\frac{1}{2}} e^{-\frac{b^2}{3}\left(\frac{1}{4}\boldsymbol{k}^2+\frac{1}{1+\varepsilon}\widetilde{\boldsymbol{q}}^2\right)} \widetilde{\mathcal{Z}}^{(1)}_{\frac{1+\varepsilon}{1+4\varepsilon}\alpha_{D_+}} \left(\frac{b|\widetilde{\boldsymbol{q}}|}{\sqrt{(1+\varepsilon)(1+4\varepsilon)}}\right) \end{cases} \ , \tag{5·69}$$

and

$$G^{LS}_{D\,d\,W}(\boldsymbol{r}, \boldsymbol{q}) = \frac{mc^2}{3b^2} \frac{\zeta+4}{2(\zeta+1)} \widetilde{\mathcal{Z}}^{(1)}_{\frac{m^2}{4\eta_d}}(m|\boldsymbol{r}|) \ ,$$

$$G^{LS}_{\binom{D_-}{D_+}d\,W} = 6\pi\hbar c b^2 \frac{\zeta+4}{2(\zeta+1)} \left(\frac{\eta_d}{\pi}\frac{1}{1+\varepsilon}\right)^{\frac{3}{2}} e^{-\eta_d \boldsymbol{r}^2}$$
$$\times \begin{cases} \frac{\eta_d}{\sqrt{1+\varepsilon}} \widetilde{\mathcal{Z}}^{(1)}_{\frac{1}{1+\varepsilon}\alpha_{D_-}} \left(\frac{1}{\sqrt{1+\varepsilon}} \frac{1}{\sqrt{3}} b|\widetilde{\boldsymbol{q}}|\right) \\ \frac{\sqrt{3}}{2} \frac{\eta_d}{1+\varepsilon} e^{-\frac{1}{1+\varepsilon}\frac{1}{3}(b\widetilde{\boldsymbol{q}})^2} \sqrt{\pi\alpha_{D_+}} \widetilde{\mathcal{Z}}^{(1)}_{(1-b^2\eta_d)\alpha_{D_+}} \left(2b\eta_d\sqrt{\alpha_{D_+}}|\boldsymbol{r}|\right) \end{cases} \ ,$$

$$G^{LS}_{D\,e\,W}(\boldsymbol{r}, \boldsymbol{q}) = \frac{\zeta+4}{2(\zeta+1)} \left(\frac{8\nu}{3\pi}\right)^{\frac{3}{2}} e^{-\frac{8\nu}{3}\boldsymbol{r}^2} \frac{8\nu}{3} \hbar c b^2 \frac{2\pi}{3} \left(\frac{1}{3\varepsilon}\right)^2 \frac{1}{\sqrt{1+4\varepsilon}} e^{-\frac{1}{4\varepsilon}\frac{4}{3}(b\widetilde{\boldsymbol{q}})^2}$$

$$\times \widetilde{\mathcal{Z}}^{(1)}_{\frac{1+4\varepsilon}{6\varepsilon}\alpha_E}\left(\frac{b|\widetilde{\boldsymbol{q}}|}{\sqrt{3\varepsilon(1+4\varepsilon)}}\right) ,$$

$$G^{LS}_{\binom{D_-}{D_+}\,e\,W}(\boldsymbol{r},\boldsymbol{q}) = 6\pi\hbar c b^2 \frac{\zeta+4}{2(\zeta+1)}\left(\frac{\eta_e}{\pi}\frac{1}{1+4\varepsilon}\right)^{\frac{3}{2}} e^{-\eta_e \boldsymbol{r}^2}$$
$$\times \begin{cases} \frac{\eta_e}{1+4\varepsilon} e^{-\frac{1}{1+4\varepsilon}\frac{4}{3}(b\widetilde{\boldsymbol{q}})^2} \sqrt{\pi\alpha_{D_-}}\, \widetilde{\mathcal{Z}}^{(1)}_{\left(1-\frac{1}{3}b^2\eta_e\right)\alpha_{D_-}}\left(\frac{2}{\sqrt{3}} b\eta_e\sqrt{\alpha_{D_-}}|\boldsymbol{r}|\right) \\ \frac{\sqrt{3}}{2}\frac{\eta_e}{(1+\varepsilon)(1+4\varepsilon)} e^{-\frac{1}{1+\varepsilon}\frac{1}{3}(b\widetilde{\boldsymbol{q}})^2}\, \widetilde{\mathcal{Z}}^{(1)}_{\frac{1+\varepsilon}{1+4\varepsilon}\alpha_{D_+}}\left(\frac{b|\widetilde{\boldsymbol{q}}|}{\sqrt{(1+\varepsilon)(1+4\varepsilon)}}\right) \end{cases} , \qquad (5\cdot70)$$

where η_d and η_e are given in Eq. (5·54). For the numerical calculation, $\widetilde{\mathcal{Z}}^{(1)}_\alpha(\rho)$ is calculated from

$$\widetilde{\mathcal{Z}}^{(1)}_\alpha(\rho) = -\frac{1}{3}\left[\frac{\rho^2}{\alpha}\widetilde{\mathcal{Z}}_\alpha(\rho) + \widetilde{\mathcal{Y}}_\alpha(\rho) - \frac{1}{2\alpha}\right] , \qquad (5\cdot71)$$

where

$$\widetilde{\mathcal{Y}}_\alpha(\rho) = \int_0^1 dt\; e^{\alpha\left(1-\frac{1}{t^2}\right)+\rho^2(1-t^2)} , \qquad \widetilde{\mathcal{Z}}^D_\alpha(\rho) = \int_0^1 t^4\; dt\; e^{\alpha\left(1-\frac{1}{t^2}\right)+\rho^2(1-t^2)} . \qquad (5\cdot72)$$

Equation (5·71) is derived from the relationship

$$\mathcal{D}_\alpha(a) = \frac{1}{2\sqrt{\pi}}\left(\frac{1}{\alpha}\right)^{\frac{3}{2}} e^{-a} = \frac{1}{\sqrt{\pi\alpha}}\int_0^1 dt\left(1+\frac{3}{2\alpha}t^2 - \frac{a}{\alpha}t^4\right) e^{\alpha\left(1-\frac{1}{t^2}\right)-at^2} . \qquad (5\cdot73)$$

In order to calculate the LS Born kernel from EMEP, we need to know the spin-flavor factors $X^{LS}_{x\mathcal{T}}$ in Eq. (5·57). There are not given in our FSS paper, Phys. Rev. C54 (1996) 2180, Appendix, since the model FSS does not include the EMEP LS force. It is not given in our fss2 paper, Phys. Rev. C65 (2002) 014002, either, since we have just dropped the Appendix for spin-flavor factors of EMEP's, at the stage of publication. We here show

Table III. The LS spin-flavor factors of the singlet and octet mesons in the operator form; $X^{\Omega}_{x\mathcal{T}}$ with $\Omega = LS$, $LS^{(-)}$, and $LS^{(-)}\sigma$. Factors for $\mathcal{T} = S$ and S' types are not listed, since their contributions are zero from the spatial integrals. Here, $X_N(S=1) = -(1/12)(e^\dagger_e e_e + e^\dagger_m e_m)$.

$(\lambda\lambda)$	$x\mathcal{T}$	LS	$LS^{(-)}$	$LS^{(-)}\sigma$
	$0D_+$	3	0	0
(00)	$1D_-$	$-\frac{1}{6}e^\dagger_e e_m$	$-\frac{1}{6}e_m e^\dagger_e$	0
	$1D_+$	$2[X_N(S=1) - X^{LS\,(00)}_{1D_-}]$	$-2X^{LS^{(-)}\,(00)}_{1D_-}$	0
	$0D_+$	${e^e_{(11)}}^\dagger e^m_{(11)}$	$e^\dagger_m e_e$	0
(11)	$1D_-$	$-2 - \frac{2}{3}X^{LS\,(00)}_{1D_-}$	$-\frac{2}{3}X^{LS^{(-)}\,(00)}_{1D_-}$	0
	$1D_+$	$X^{LS\,(00)}_{1D_+} 2\left(\mathcal{P} - \frac{1}{3}\right)$	$-\frac{2}{3}X^{LS^{(-)}\,(00)}_{1D_+}$	$X^{LS^{(-)}\,(00)}_{1D_+} 2\mathcal{P}$

only the result for the ΛN diagonal channels in various stages. The first stage is the factors in the SU_3 limit. This implies the flavor-singlet and octet scalar mesons (S: ϵ, S^*, δ, and κ) and the corresponding vector mesons (V: ω, ϕ, ρ, K^*). In the operator form, these are given as in Table III. Note that, although there exists a $LS^{(-)}\sigma$ component in Table III, we have no $LS^{(-)}\sigma$ force in fss2 since we have neglected the mass difference of M_N and M_Y in the EMEP's. (See Appendix A of our Extended Time Reversal Paper, Prog. Theor. Phys. 97 (1997) 587.) The explicit values in the ΛN - ΛN diagonal channel are calculated from

$$
\begin{aligned}
& e^e_{(00)} = -\sqrt{6} \ , \qquad e^m_{(00)} = -\frac{\sqrt{6}}{5}\frac{5}{3} \ , \\
& e^\dagger_e e_m = {e^e_{(00)}}^\dagger e^m_{(00)} + {e^e_{(11)}}^\dagger e^m_{(11)} = 2 + {e^e_{(11)}}^\dagger e^m_{(11)} \ , \\
& {e^e_{(11)}}^\dagger e^m_{(11)} = {e^m_{(11)}}^\dagger e^e_{(11)} = -1 + 3\mathcal{P} \ , \qquad e^\dagger_e e_m = 1 + 3\mathcal{P} \ , \\
& e^\dagger_e e_e = 6 + {e^e_{(11)}}^\dagger e^e_{(11)} = 6 + 3\mathcal{P} \ , \qquad e^\dagger_m e_m = 3\mathcal{P} \ , \\
& X_N(S=1) = -\frac{1}{2}(1+\mathcal{P}) \ ,
\end{aligned}
\tag{5·74}
$$

for $\Omega = LS$ with $\mathcal{P} = \mathcal{P}'$. On the other hand, $\Omega = LS^{(-)}$ and $LS^{(-)}\sigma$ terms correspond to $\mathcal{P} \neq \mathcal{P}'$, and we obtain $e_m e^\dagger_e = -e^\dagger_e e_m = -1$ from the formula in Eq. (A.11) of Prog. Theor. Phys. Suppl. No. 91 (1987) 160. The results are tabulated in Table IV.

The spin-flavor factors with the full flavor symmetry breaking included can be read from the Maple program, which are included in the Appendix C of the original version of our fss2 paper. The $0D_+$, $1D_-$, and $1D_+$ factors of the LS type in the ΛN diagonal channels are listed in Table V. From the isoscalar factors with $I = 0$, we obtain

$$
X^{\epsilon_1\,\Omega}_{x\mathcal{T}} = X^{I=0\,\Omega}_{x\mathcal{T}}(\lambda = 1) \ , \qquad X^{S^*_8\,\Omega}_{x\mathcal{T}} = \frac{1}{3} X^{I=0\,\Omega}_{x\mathcal{T}}\left(\lambda = -\frac{1}{2}\right) \ . \tag{5·75}
$$

We can also check the sum rule of the flavor octet contributions by

$$
X^{\Omega\,(11)}_{x\mathcal{T}} = X^{\epsilon_1\,\Omega}_{x\mathcal{T}} + X^{\delta\,\Omega}_{x\mathcal{T}} + X^{\kappa\,\Omega}_{x\mathcal{T}} \ , \tag{5·76}
$$

Table IV. The ΛN - ΛN matrix elements of the LS spin-flavor factors in Table III.

$(\lambda\lambda)$	$x\mathcal{T}$	LS	$LS^{(-)}$	$LS^{(-)}\sigma$
	$0D_+$	3	0	0
(00)	$1D_-$	$-\frac{1}{2}\left(\frac{1}{3}+\mathcal{P}\right)$	$\frac{1}{6}$	0
	$1D_+$	$-\frac{2}{3}$	$-\frac{1}{3}$	0
	$0D_+$	$-1+3\mathcal{P}$	-1	0
(11)	$1D_-$	$-\frac{1}{3}\left(\frac{17}{3}-\mathcal{P}\right)$	$-\frac{1}{9}$	0
	$1D_+$	$\frac{4}{3}\left(\frac{1}{3}-\mathcal{P}\right)$	$\frac{2}{9}$	$-\frac{2}{3}\mathcal{P}$

Table V. The ΛN - ΛN matrix elements of the LS spin-flavor factors with the flavor symmetry breaking for each isospin I state. Here, $\lambda = (m_s/m_{ud}) > 1$.

I	$x\mathcal{T}$	LS	$LS^{(-)}$	$LS^{(-)}\sigma$
	$0D_+$	$1+\frac{2}{\lambda}$	$-1+\frac{1}{\lambda}$	0
0	$1D_-$	$-\frac{1}{2}\left(\frac{1}{3}+\frac{1}{\lambda}\mathcal{P}\right)$	$\frac{1}{6}$	0
	$1D_+$	$-\frac{2}{3}\frac{1}{\lambda}$	$-\frac{1}{3}\frac{1}{\lambda}$	0
	$0D_+$	0	0	0
1	$1D_-$	$-\frac{1}{2}$	$\frac{1}{2}$	0
	$1D_+$	0	0	0
	$0D_+$	$3\mathcal{P}$	0	0
1/2	$1D_-$	$-\frac{4}{3}$	$-\frac{2}{3}$	0
	$1D_+$	$-\frac{4}{3}\mathcal{P}$	0	$-\frac{2}{3}\mathcal{P}$

where δ and κ imply the $I=1$ and $I=1/2$ factors in Table V, respectively. For the discussion of the singlet-octet meson mixing, see the spin-flavor factor note, YN-Part XII, Section 6, page 64. We define the qqM coupling constants

$$f_i(\theta) = f_1\cos\theta - f_8\sin\theta\lambda_{8i} \ , \qquad g_j(\theta) = g_1\cos\theta - g_8\sin\theta\lambda_{8j} \ , \tag{5·77}$$

where θ is the mixing angle and λ_{8i} is the 8-th component of the Gell-Mann matrix for the i-th quark. The RGM spin-flavor matrix elements for the product, $f_i(\theta)g_j(\theta)$, with the spin operator are expressed as

$$\begin{aligned} W^{\Omega}_{x\mathcal{T}}(\theta) = {} & f_1g_1(\cos\theta)^2 X^{\eta_1\,\Omega}_{x\mathcal{T}} + f_8g_8(\sin\theta)^2 X^{\eta_8\,\Omega}_{x\mathcal{T}} \\ & - \left(f_8g_1 X^{\eta_8 1\,\Omega}_{x\mathcal{T}} + f_1g_8 X^{1\eta_8\,\Omega}_{x\mathcal{T}}\right)\cos\theta\sin\theta \ . \end{aligned} \tag{5·78}$$

(Note that η_1 and η_8 in the superscripts only specify the flavor degree of freedom (by using the names of PS mesons).) From Eq. (5·78), we can easily show

$$W^{\Omega}_{x\mathcal{T}}(\theta) + W^{\Omega}_{x\mathcal{T}}(\pi-\theta) = f_1g_1 X^{\eta_1\,\Omega}_{x\mathcal{T}} + f_8g_8 X^{\eta_8\,\Omega}_{x\mathcal{T}} \ . \tag{5·79}$$

In principle, we have to calculate Eq. (5·78) for each of the $x\mathcal{T}$ types. This is, however, not necessary for deriving $\Lambda\alpha$ spin-flavor factors due to the spin-isospin saturated nature of the α-cluster. To show this, we first add up all three LS-type factors for each $x\mathcal{T}$. We find

$$X^{LS\,d}_{x\mathcal{T}} + X^{LS\,e}_{x\mathcal{T}}\mathcal{P} = 4\left[X^{LS}_{x\mathcal{T}} + X^{LS^{(-)}}_{x\mathcal{T}} + X^{LS^{(-)}\sigma}_{x\mathcal{T}}\right] \ , \tag{5·80}$$

for each meson contribution. This relationship is valid since the $LS^{(-)}$ factors in Table V have no strange part (with $\mathcal{P}$) and the $LS^{(-)}\sigma$ factors no up-down part. This can be proved in general, as in Eqs. (5.12) and (5.16) of our Extended Time Reversal paper (Prog. Theor. Phys. 97 (1997) 587), based on the symmetry discussion. From the factors in Table V, we immediately realize that there is no contribution for the isovector mesons ($I = 1$) (since this is the ΛN system), and that the factors with different $x\mathcal{T}$ types have very simple ratios as in Table VI. This discussion is also applicable to the singlet-octet meson mixing, since the $1/\lambda$-dependence in the YN interaction is linear as is seen in Table V. We therefore only need to calculate the factors, $X^{(I=0)\,d}_{0D_+}(\theta)$ and $X^{(I=1/2)\,e}_{0D_+}(\theta)$. The final result is

$$
\begin{aligned}
&X^{(I=0)\,d}_{0D_+}(\theta) = (-6)X^{(I=0)\,e}_{1D_-}(\theta) = (-3)X^{(I=0)\,d}_{1D_+}(\theta) \\
&= 4\left[3f_1g_1(\cos\theta)^2 - 2f_8g_8(\sin\theta)^2 + \frac{\sqrt{3}}{2}\left(f_1g_8 + f_8g_1\right)\cos\theta\sin\theta\right] \ , \\
&X^{(I=1/2)\,e}_{0D_+}(\theta) = \left(-\frac{3}{2}\right)X^{(I=1/2)\,d}_{1D_-}(\theta) = \left(-\frac{3}{2}\right)X^{(I=1/2)\,e}_{1D_+}(\theta) = 4\cdot 3f_8g_8 \ . \qquad (5{\cdot}81)
\end{aligned}
$$

The coupling constants, f_1, f_8, g_1, g_8 are for qqM coupling. These are expressed by the standard baryon-meson coupling constants give in the FSS and fss2 papers through the following relationship:

$$
\begin{aligned}
&f_1 = g_1 = \frac{1}{3}f_1^{\rm S} \ , \qquad f_8 = g_8 = f_8^{\rm S} \qquad \text{for S-mesons} \ , \\
&f_1 = f_1^{{\rm V}m} \ , \quad g_1 = \frac{1}{3}f_1^{{\rm V}e} \ , \quad f_8 = \frac{3}{5}f_8^{{\rm V}m} \ , \quad g_8 = f_8^{{\rm V}e} \qquad \text{for V-mesons} \ . \qquad (5{\cdot}82)
\end{aligned}
$$

Finally, the EMEP contribution to the LS Born kernel $F(q_f, q_i, \theta)$ in Eq. (5·13) is given by

$$
F(q_f, q_i, \theta) = -\sum_\beta \begin{pmatrix} (b\gamma)^2 \\ 4b^2\gamma \end{pmatrix} \left[X^{\beta\,LS\,d}_{0D_+}V^{LS}_d(\boldsymbol{q}_f, \boldsymbol{q}_i) + X^{\beta\,LS\,e}_{0D_+}V^{LS}_e(\boldsymbol{q}_f, \boldsymbol{q}_i)\right] \ , \quad (5{\cdot}83)
$$

Table VI. The $\Lambda\alpha$ spin-flavor factors derived from Table V. The contributions from the isovector mesons with $I = 1$ are zero.

	$I=0$			$I=1/2$		
	$X^{LS\,d}_{x\mathcal{T}}(\lambda)/4$	$X^{LS\,e}_{x\mathcal{T}}(\lambda)/4$	ratio	$X^{LS\,d}_{x\mathcal{T}}/4$	$X^{LS\,e}_{x\mathcal{T}}/4$	ratio
$0D_+$	$1+\frac{3}{\lambda}$	0	1	0	3	1
$1D_-$	0	$-\frac{1}{2\lambda}$	$-\frac{1}{6}$	-2	0	$-\frac{2}{3}$
$1D_+$	$-\frac{1}{\lambda}$	0	$-\frac{1}{3}$	0	-2	$-\frac{2}{3}$

where the spatial functions are given by

$$V_d^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i) = V_{D\,d}^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i) - \frac{1}{3}V_{D_+\,d}^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i) + \frac{1}{6}V_{D_-\,e}^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i) \ ,$$
$$V_e^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i) = V_{D\,e}^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i) - \frac{2}{3}V_{D_+\,e}^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i) + \frac{2}{3}V_{D_-\,d}^{LS}(\boldsymbol{q}_f,\boldsymbol{q}_i) \ , \qquad (5{\cdot}84)$$

in terms of the spatial integrals in Eq. (5·69). Here, $\gamma = (m/2m_{ud})$ with m being the meson mass. The $\Lambda\alpha$ Wigner transform $G_W^{LS}(\boldsymbol{r},\boldsymbol{q})$ is given by the same expression, using

$$G_{d\,W}^{LS}(\boldsymbol{r},\boldsymbol{q}) = G_{D\,d\,W}^{LS}(\boldsymbol{r},\boldsymbol{q}) - \frac{1}{3}G_{D_+\,d\,W}^{LS}(\boldsymbol{r},\boldsymbol{q}) + \frac{1}{6}G_{D_-\,e\,W}^{LS}(\boldsymbol{r},\boldsymbol{q}) \ ,$$
$$G_{e\,W}^{LS}(\boldsymbol{r},\boldsymbol{q}) = G_{D\,e\,W}^{LS}(\boldsymbol{r},\boldsymbol{q}) - \frac{2}{3}G_{D_+\,e\,W}^{LS}(\boldsymbol{r},\boldsymbol{q}) + \frac{2}{3}G_{D_-\,d\,W}^{LS}(\boldsymbol{r},\boldsymbol{q}) \ , \qquad (5{\cdot}85)$$

in terms of Eq. (5·70). We can also calculate the EMEP contribution to the Scheerbaum factor S_Λ in the Born approximation, using the $\boldsymbol{P} = 0$ Wigner transform, once the $\Lambda\alpha$ spin-flavor factors are obtained. This is given by

$$S_\Lambda = -\hbar c b^4 \frac{1}{1+\zeta}\sum_\beta \begin{pmatrix} \gamma^2 \\ 4\gamma \end{pmatrix}\left[X_{0D_+}^{\beta\,LS\,d} f_d + X_{0D_+}^{\beta\,LS\,e} f_e\right] \ , \qquad (5{\cdot}86)$$

where

$$f_d = f_D - \frac{1}{3}f_{D_+} + \frac{1}{6}f_{D_-} \quad , \qquad f_e = f_D - \frac{2}{3}f_{D_+} + \frac{2}{3}f_{D_-} \ , \qquad (5{\cdot}87)$$

with

$$f_D = \frac{1}{3\alpha_E}\frac{1}{(b\boldsymbol{k})^2 + 2\alpha_E}e^{-\frac{1}{3}(b\boldsymbol{k})^2} \ ,$$
$$f_{D_-} = \frac{3}{2}e^{-\frac{1}{3}(b\boldsymbol{k})^2}\widetilde{\mathcal{Z}}_{\alpha_{D_-}}^{(1)}(0) \ ,$$
$$f_{D_+} = \frac{3}{2}\left(\frac{3}{4}\right)^{\frac{1}{2}} e^{-\frac{1}{12}(b\boldsymbol{k})^2}\widetilde{\mathcal{Z}}_{\alpha_{D_+}}^{(1)}\left(\frac{1}{2}b|\boldsymbol{k}|\right) \ . \qquad (5{\cdot}88)$$

The value of $\widetilde{\mathcal{Z}}_\alpha^{(1)}(0)$ is given by

$$\widetilde{\mathcal{Z}}_\alpha^{(1)}(0) = \sqrt{\pi\alpha}Z_\alpha^{(1)}(0) \ ,$$
$$Z_\alpha^{(1)}(0) = \frac{1}{3}\left[e^\alpha \mathrm{erfc}(\sqrt{\alpha}) + \frac{1}{\sqrt{\pi\alpha}}\left(\frac{1}{2\alpha} - 1\right)\right] \ . \qquad (5{\cdot}89)$$